Cogn Process
DOI 10.1007/s10339-012-0445-0

SHORT REPORT

How do spatial representations enhance cognitive numerical processing?

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Abstract Several philosophical theories attempt to explain how actions performed in the world enhance cognitive processing: internalism, active externalism, and cognitive integration. The aim of this paper is to examine whether the use of spatial representations in arithmetic can shed light on this debate. Relying on philosophical analysis, on a discussion of empirical work in the cognitive neuroscience of number, and on a historical case study, I will show that spatial representations of number indicate an integration between internal and external cognitive processes.

Keywords Spatial representations · Numerical cognition · Cognitive integration · Chinese algebra

Introduction

When engaged in cognitive tasks, humans often perform actions that involve the manipulation of material objects, for instance, a scrabble player who rearranges tiles on her tray, an engineer who draws a diagram, or a child who uses her fingers to count. How exactly these actions enhance cognition is a matter of debate in the philosophy of cognitive science. According to internalism (e.g., Adams and Aizawa 2001), performing actions in the external world can enhance and improve cognitive performance, but such actions are not themselves part of cognition. Cognitive processes only take place inside the skull. For instance, when one uses an abacus to solve an arithmetical task, the only action that is cognitive is the retrieval from memory of abacus techniques and interpreting the result by converting the observed configuration of beads into an internal representation of mental magnitude.

Active externalists (e.g., Clark and Chalmers 1998) find internalism too limiting and argue that in many cases, external tools are part of cognitive processes. However, if one simply grants cognitive status to every object that is somehow causally involved in cognitive processes, we end up with sentient pencils and notepads, because they happen to be used in pen-and-paper calculations. To mitigate this problem of cognitive bloat, Clark and Chalmers (1998) propose the parity principle, which holds that if we characterize a process that takes place in the brain as cognitive, we also ought to characterize a structurally similar process that takes place outside of the brain as such. Accordingly, the notebook with appointments and facts, used by an Alzheimer's patient, is part of his cognition, because he uses it in a way that is functionally isomorphic to memories in a neurologically normal person.

Proponents of cognitive integration (e.g., Menary 2007) also propose that actions involving external objects can be part of cognition, but unlike externalists, maintain that such actions are often structurally different from internal cognitive processes. Cognition often integrates internal and external components, and both types of processes influence each other. The Google effect illustrates that the widespread use of search engines and digital storage is altering the way humans use their natural memory. In recalling factual information, we now tend to rely on data retrieved

Electronic supplementary material The online version of this article (doi:10.1007/s10339-012-0445-0) contains supplementary material, which is available to authorized users.

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Published online: 18 July 2012

online (external memory), tagged by an internally stored memory of our search query, for example, what keywords we used when we found the information, or which websites we visited (Sparrow et al. 2011). Semantic memory has now shifted from an internal storage of factual information to an internal storage of where it can be retrieved, crucially dependent on external devices like computers and iPhones.

Understanding spatial representations in mathematical practice

Humans recruit a variety of external spatial representations when they perform arithmetical operations. It has been widely recognized that these representations enhance our effectiveness in solving numerical problems, but it remains unclear how this relationship can be understood. In the remainder of the paper, I will examine which of the three philosophical models of externalized cognition can best describe the use of spatial representations in numerical cognition.

Internalism

Across cultures, humans spontaneously make associations between numbers and space. Although there is a wide diversity in how this is implemented (for instance, numbers can be mapped onto body parts, abacuses or rulers), space–number association appears to be a human universal. Even people from cultures without precise number words, such as the Amazonian Mundurucú, spontaneously map numbers onto a logarithmic mental number line (Dehaene et al. 2008). According to internalism, this is a result of an internal cognitive association between numbers and space. At the neural level, representations of space and number overlap to some extent, for example, populations of neurons in the intraparietal sulci of rhesus monkeys (Tudusciuc and Nieder 2007) that respond to number and space (e.g., line length) partially overlap.

Internalism predicts that numerical tasks will be converted into an inner abstract, stimulus-independent code; thus, the format of the task will not make much difference to how the task is solved. In philosophy of mathematics, this position is termed presentism. According to presentists (e.g., Barabashev 1997), one can translate any mathematical idea expressed in a foreign medium (e.g., body-part counting, Roman numerals) into present-day Western mathematical notation without changing the content. Since external representations can at most facilitate internal processing but do not alter it in other ways, the only difference one might observe are variations in speed of computation, for example, a multiplication in Roman numerals takes longer to solve than one in Arabic digits. However, several studies indicate that differences in format and modality (e.g., numbers presented as dots or digits) do influence cognitive processing, also at the neural level (see Cohen Kadosh and Walsh 2009, for a review). Although the space–number association has an internal component, internalism cannot adequately describe how numerical and spatial representations interact.

Active externalism

To develop the active externalist case for numbers, one would have to show that spatial representations of numbers are structurally similar to internal cognitive representations, following the parity principle. The cognitive science literature does not agree on the format of mental magnitudes (see De Cruz and De Smedt 2010, for a review). Some authors have argued that numerical representations are linear and serial, whereas others regard them as parallel and logarithmic. Yet others conceptualize mental magnitudes as essentially non-spatial, namely as slots of working memory (so-called object files), which represent small sets of objects. Regardless of which model one adheres to, there is no parity between external numerical representations and internal magnitudes. At present, no active externalist case can be made for space–number associations.

Cognitive integration

According to cognitive integration, the internal and external parts of numerical cognition are complementary, and need not be isomorphic. It predicts a causal, dynamic interaction between both types of processes. Internal cognitive processes constrain the development and cultural transmission of external numerical representations. For instance, although it is logically possible to have numerical notation systems with very large or very small base sizes, few cultures with numerical notation systems have developed them (e.g., the binary system is culturally exceptional), because there is a trade-off between working memory and ease of processing. Conversely, the choice of a particular external spatial representation can influence the internal processes involved in arithmetical cognition. Western children, for example, gradually develop a linear mental representation of number, which seems to be facilitated by engaging with external spatial representations: Siegler and Ramani (2008) showed that playing numerical board games improved the linear representation of number in preschoolers from low socioeconomic backgrounds. The next section will explore this interaction between internal and external cognition in more detail, through the case study of classical Chinese linear algebra.

A case study of cognitive integration: Chinese algebra

Chinese linear algebra emerged during the Qín and Hàn dynasties, from ca. 221 BCE to 220 CE (Hart 2010). The nascent Chinese empire was faced with practical problems, such as land surveying for large-scale public works and taxation. It employed professional mathematicians, *fāngchēng* practitioners; the term *fāngchēng* means measurement by placement [of rods] side by side. This method was used to solve arithmetical problems, by laying out counting rods in a grid structure on a mat (see Fig. 1). Problems involving simultaneous linear equations appear in the anonymous *The nine chapters on the mathematical arts* (*Jiǔzhāng Suàn shù*), a compilation dated to the first century CE that has been extensively studied in imperial Chinese history (Chemla and Guo 2004). Counting rods were arranged in rows and columns, where each row corresponded to the coefficient of an unknown and each column to an equation. *Fāngchēng* practitioners solved these equations by multiplying rows with each other and subtracting the result term-by-term to eliminate cells in the matrix, a procedure that bears remarkable similarities to the method of Gaussian elimination, developed in nineteenth-century Europe by Gauss (foreshadowed by Newton). Consider the following problem from *The nine chapters* (chapter 8, problem 3):

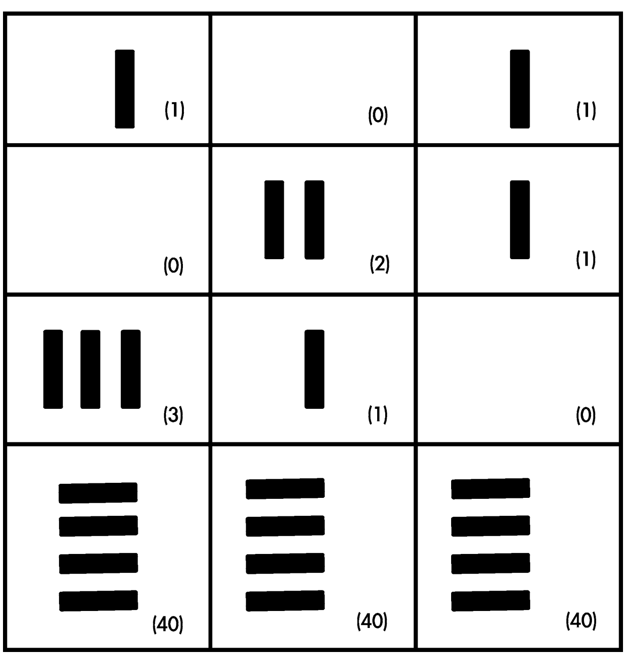

Fig. 1 The *fāngchēng* representation of problem 8.3. (Quantities are put in Arabic digits between brackets for clarification.) All rods on this figure would actually be red, since all terms are positive. One can turn the Chinese notation 90° counterclockwise to obtain the modern matrix notation devised by Gauss

Given one warhorse, two common horses, and three inferior horses that are grouped into teams that collectively can transport 40 *dan* [ca.1,200 kg]. Teams are composed of 1 warhorse and 1 common horse, 2 common horses and 1 inferior horse, 3 inferior horses and 1 warhorse. How much weight can a warhorse, common horse, and inferior horse carry? Figure 1 shows the problem in *fāngchēng* notation.

In modern algebraic notation this is

$$\begin{cases} x + y + 0 = 40 \\ 0 + 2y + z = 40 \\ x + 0 + 3z = 40 \end{cases}$$

The procedure is shown in the supplementary movie S1. *Fāngchēng* is an integration of external and internal cognitive processes. The external part consists of the representation of the base values of numerals (number of rods), the power values (horizontal or vertical positioning of rods), and the sign of numerals (color of rods). Variables and unknowns were represented by their position in the matrix, obviating the need for symbols to denote them. Zero was simply represented as an empty space and thus did not require a separate symbol. Negative numbers were denoted by black rods, positive numbers by red rods. Thus, if during the term-by-term subtraction, a negative number arose, the cluster of red rods was replaced by one or more black rods, as can be seen in S1. This clear visual difference between positive and negative numbers decreased the chance of error that results from forgetting to carry a minus sign. The internal component of *fāngchēng* consists of remembering and correctly following the rules for the manipulation of the rods, that is, the order in which the various operations (cross-multiplications, term-by-term subtractions) are to be performed. Also, throughout, the *fāngchēng* practitioner needs to keep track of which calculations have already been performed, by observing which cells are already empty, that is, eliminated.

The advantages of this system are multiple: quantities are easily visualized, mistakes with negative numbers are less likely, and the matrix needs not be redrawn each time (as in Gaussian matrices with Arabic numerals), saving time and avoiding copying errors. Also, the classical European method of determinants,[1] developed by Leibniz and others, has a much steeper rise in number of operations as *n* increases, compared to the Chinese (and Gaussian) method. For instance, chapter 8 of *The nine chapters* contains equations with up to 5 unknowns, which are still manageable within the Chinese system, requiring 175 operations, but unmanageable in the method of determinants, requiring as many as 5039 operations (Hart 2010).

[1] This method is an alternative way of solving simultaneous linear equations, predating Gaussian elimination.

Since Arabic digits are not as well suited for the matrix method as counting rods (crucially, one cannot move them freely as one solves the matrix), it is no coincidence that Western European mathematicians only came up with a similar solution in the early nineteenth century. The choice of the medium (counting rods on a mat) has thus allowed for the early development of matrix methods in Chinese algebra. At the same time, internal cognitive processes influenced how the method developed, for example, in the base size of the numerals (a combination of base-5 and base-10). These inextricable causal links between external and internal cognitive processes are diagnostic of cognitive integration.

However, *fāngchēng* also had its disadvantages. If one makes a mistake, it is difficult to track down where it occurred since there is no notational record of the individual operations. As the third-century mathematician Liú Huī remarked "Those individuals who do not understand the subtleties of the [*fāngchēng*] method just use the procedure to fill a whole felt carpet with rods, but since they rejoice only in complexity, they delight in making mistakes" (Liú Huī cited in Chemla and Guo 2004, commentary on problem 8.18, p. 651). Another limitation of this system was that it had no separate symbols for arbitrary unknowns and variables: In imperial China, mathematical solutions were always tied to particular exemplars, involving concrete situations and quantities. As a result, Chinese proofs were rare, and most were visual proofs. The external format of the *fāngchēng* procedure was less suitable for proofs than European algebra, which has symbols that can express arbitrary variables (De Cruz and De Smedt in press).

Concluding remarks

The case study of *fāngchēng* indicates that spatial representation can enhance numerical cognition. The use of counting rods by Chinese *fāngchēng* practitioners does not seem to be a mere translation of internal cognitive processes (as internalism holds) nor is it isomorphic to how the brain computes numbers (as active externalism proposes). Rather, in this historical method, there is a dynamic relationship between internal and external parts of mathematical practice. This study tentatively suggests a causal link between the use of counting rods and the invention of matrix methods early in Chinese history.

Conflict of interest This supplement was not sponsored by outside commercial interests. It was funded entirely by ECONA, Via dei Marsi, 78, 00185 Roma, Italy.

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