HOW DOES COMPLEX MATHEMATICAL THEORY ARISE? PHYLOGENETIC AND CULTURAL ORIGINS OF ALGEBRA

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Algebra has emergent properties that are neither found in the cultural context in which mathematicians work, nor in the evolved cognitive abilities for mathematical thought that enable it. In this paper, I argue that an externalization of mathematical operations in a consistent symbolic notation system is a prerequisite for these emergent properties. In particular, externalism allows mathematicians to perform operations that would be impossible in the mind alone. By comparing the development of algebra in three distinct historical cultural settings—China, the medieval Islamic world and early modern Europe—I demonstrate that such an active externalism requires specific cultural conditions, including a metaphysical view of the world compatible with science, a notation system that enables the symbolic notation of operations, and the ontological viewpoint that mathematics is a human endeavour. I discuss how extending mathematical operations from the brain into the world gives algebra a degree of autonomy that is impossible to achieve were it performed in the mind alone.

1. What Would Mathematics without Culture Look Like?

Even a concept as seemingly simple as the number two is a highly abstract representation. There is no obvious perceptual similarity between two cows and two bicycles, and yet both collections share the concept of 'twoness'. A large number of experimental studies nevertheless indicate that both non-human animals and human infants have a robust understanding of number (see Ref. 1 for an overview). For example, when presented with two different numerosities of food-items, red-backed salamanders go for the larger quantity [2]. Human infants, likewise, can discriminate between small and larger collections (e.g. two versus three, eight versus sixteen)—even when non-numerical variables such as cumulative surface area are controlled for [3]. They can solve simple addition and subtraction problems involving both small and larger quantities [4]. Recently, single cell recordings in rhesus monkeys [5] have identified number-sensitive neurons—individual neurons that respond only to changes in number, while remaining insensitive to changes in shape or size. Each neuron is tuned to a preferred quantity: a neuron preferentially firing at, say two, will fire a bit less at one or three, and even less when observing higher quantities. These neural tuning curves become broader as quantities increase. Numerosities are thus not represented as a linear mental number line, but more as a logarithmic ruler, in which the psychological distance between one and two is considerably greater than between fifty and one hundred.

Yet, mathematical systems without exact number representations can only capture the most rudimentary of numerical relationships. Evidence for this claim comes from two Amazonian cultures, the Pirahã [6] and the Mundurukú [7], where exact number words do not exist, such that the word for 'one' can mean 'two' and vice versa. People in these cultures cannot even discriminate a box with three fish from a box with four fish painted on it. They do not possess a counting routine or any other cultural tool which helps them to construct positive integer values. Conversely, despite extensive training, no non-human animal has yet been able to learn complex mathematical operations, such as exact positive integer representations. One long-term training programme extending over twenty years involved teaching Ai, a female chimpanzee, to understand and produce Arabic digits [8]. Ai never managed to count more than nine items, and never generalized to the counting procedure that children master with ease. These lines of evidence combined suggest that although our mathematical abilities build upon an evolved number sense which we share with other animals, they are clearly more than that. It is as yet unclear how this cognitive adaptation can account for the vast proliferation and complexity of cultural mathematical concepts.

In this paper, I will argue that complex mathematical theory can emerge if humans extend their minds into a symbolic notation system. Using algebra as a case-study, I will demonstrate how humans can overcome their cognitive limitations by externalizing operations that are difficult or impossible to perform in the mind alone. I begin by outlining which cognitive processes are involved when people learn and perform algebra. I show how humans combine several evolved specialized neural circuits to solve equations, but that learning algebra still requires synaptic rewiring. I then go on to explain how cultural factors, including religion, symbolic notation and philosophy influence the development and level of abstraction in algebraic systems in China, the Islamic world and Europe. I argue that this development critically depends upon the elaboration of a symbolic notation system, and the assumption that mathematics is a human endeavour, which can be improved by individual mathematicians. Finally, I discuss how mathematics can have emergent properties as a result of this externalism. Mathematics may be a hybrid system of knowledge, in that it contains operations that are performable in the head, and operations that can only exist as symbols.

2. Cognitive Processes Involved in Learning and Performing Algebra

Classical algebra is a set of procedures and rules for solving equations with one or more unknowns. How does one acquire this procedural knowledge? It is important to note that the transmission of a concept such as 'democracy' does not entail that this concept is downloaded from one brain to another: rather, when I hear about democracy I reconstruct it in my own mind, and it is quite likely that internal representations of democracy are not identical [9]. Some concepts are more easily transmitted than others because they require minimal cultural input—in such cases the rest of the concept is 'filled in' by the recipient's tacit assumptions. For example, someone who learns that a zombie does not move according to his own free will tacitly assumes that such a being does obey the laws of physics. Thus, only the counterintuitive part of this concept—a violation of intuitions about agency—needs to be transmitted [10]. Given the minimal transmission requirements for such religious concepts, it is not surprising that they should have evolved many times independently, occurring throughout a wide array of physical and cultural settings. Neither their birth nor persistence seems to require exceptional cultural conditions [11].

In stark contrast to religious concepts, algebra is non-intuitive: it requires a co-optation of evolved cognitive strategies which are normally not deployed together. This results in a new system with emergent properties not present in any of the subsystems. We can observe this in other domains as well: hunter-gatherers for instance, track down animals using a combination of several domains of knowledge outside of biology, including physics (like examining spoor and damage the animal caused to plants) and psychology (imaging what one would do if one were in the position of the prey) [12]. As a result, the emergent properties of tracking enable the hunter to work out solutions which would have been impossible in each of the subsystems. The drawback for these enhanced cognitive abilities is that such skills are typically hard to learn. Male hunter-gatherers start learning to track when they are adolescents, but only gain expertise—as measured by their return-rates—when they are well in their thirties [13].

Likewise, fMRI studies (e.g. Ref. 14) found that people use different brain circuits when they solve equations (Fig. 1):

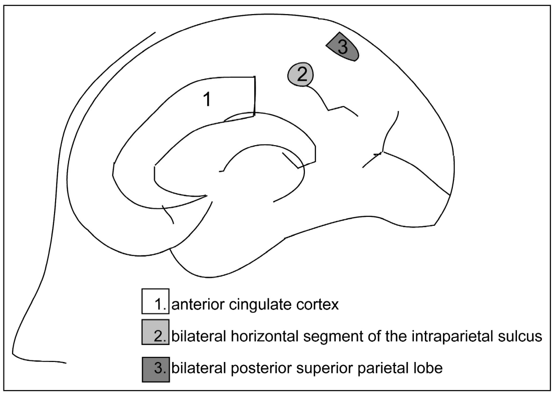

Figure 1. Brain areas involved in solving equations (only left hemisphere shown)

- the anterior cingulate cortex, which is otherwise active when people reflect on other people's mental states [15]
- the intraparietal sulci, which have been implicated in brain imaging studies involving number [16]
- the posterior parietal cortex, which is normally activated in visuospatial tasks, including spatial working memory and attention orienting [17].

One study investigating the difference between mathematical strategies in normal and in precocious teenagers found that the latter deployed spatial skills to solve algebraic problems by diagramming important relationships apparent in the problem [18]. Thus, solving equations depends on a successful co-optation of several brain areas which are normally involved in ecologically relevant tasks such as detecting numerosities and assessing spatial relationships. Perhaps the most surprising finding is that after a learning period adolescents exhibited alterations in their cerebral blood flow when solving equations [19]. In other words, the brain rewires its synaptic connections in response to the task-demands. Thus, we cannot explain cultural mathematical concepts solely as a result of the combination of specialized neural circuits, since learning these produces permanent synaptic changes in a few well-restricted areas of the brain.

3. Cultural Transmission of Algebra in Context

Like other domains of human culture, algebra has emergent properties which cannot be traced back to the innate cognitive architecture of our brains. Human culture is cumulative, unlike social traditions in non-human species. Our cultural systems build and elaborate upon the inventions of previous generations, which has resulted in a vast proliferation of artefact types, customs, religious beliefs and social systems that vary across cultures. Why only humans can benefit from this ratchet effect of cultural evolution is not entirely clear. One proposal, the extended mind thesis, formulated by Clark and Chalmers [20] is that humans extend thought processes into the external world. External memory devices such as books or electronically stored documents, instruments such as computers or nautical slide rulers, or simply pen and paper, all serve to extend computations beyond the brain. Clearly, the use of external media enables us to accumulate information beyond the scope of the individual memory. Moreover, working with external symbolic media may constitute an epistemic act in itself. Some actions performed by use of external media solve problems more easily and reliably than if they had been solved in the mind alone. Take the computer game Tetris: physically rotating the two-dimensional blocks by means of a keyboard in order to fit them in the slots proves far more efficient than mental rotation [21]. Note that, in all these cases, cognition only takes place within the human brain [22].

The use of external media makes computational solutions possible that could not have been reached without them. In mathematics, a positional system such as the Hindu numerals renders multiplications with large numbers easy and transparent, while multiplication in non-positional systems such as the Roman numerals is a far more daunting task. As I shall demonstrate, the absence or presence of symbols that represent operations influences the degree of abstraction within specific mathematical systems. Only in early modern Europe did various cultural conditions shape the possibility of a consistent externalism of mathematical ideas.

3.1. *Chinese Algebra*

Chinese algebra first bloomed around the beginning of the Common Era with the anonymous Nine Chapters on the Mathematical Arts (*Chiu Chang Suan Shu*). It was organized around practical problems of public services, such as taxation and digging canals [23]. During the T'ang dynasty (618–906), China exhibited an unusual openness to foreign influences, welcoming Arab and Indian scholars. Thanks to this influx, by the end of the Sung dynasty (900–1279), Chinese algebra had attained a level that would only be surpassed in Europe in the course of the eighteenth century. During the Ming dynasty (1368–1644), as a reaction to the Mongol invasions, indigenous culture and values were restored. This led to a stagnation and decline in all sciences. Finally, during the Qing dynasty (1644–1911), there was an increase in influence from European mathematics, which led to the demise of Chinese algebra and other mathematical systems [24]. Why did Chinese algebra come to a standstill, and why was it eventually abandoned in favour of Western algebra? A possible factor is the notation system, shown in Fig. 2. To represent a system of equations, counting rods were arranged in such a way that each column was assigned to each equation, and each row corresponded to the coefficients of each unknown. This in turn promoted the invention of matrix methods to solve simultaneous linear equations and higher-order equations. Figure 2, for example, represents the following equations:

$$\begin{cases} 2x - 3y + 8z = 32 \\ -6x - 2y - z = 62 \\ 3x + 21y - 3z = 0 \end{cases}$$

However, these rods were less useful to express general abstract rules other than actual calculations, which preserved the concreteness of Chinese mathematics [23]. Consequently, Chinese algebra textbooks never attempted to give an abstract formulation of a general rule, but presented examples that served as paradigms to solve similar problems [25]. Next to this, the venerable status of the Nine Chapters on the Mathematical Arts impeded further progress. Numerous mathematicians wrote commentaries on it, and as time went by commentaries on those commentaries, and even those who produced original work felt obliged to extensively refer to it. This attitude stems from the Confucian point of view that only the wise sages of the past have attained true wisdom; it was the duty of aspiring scholars to emulate their mental state [26]. The high status of mathematicians gradually eroded as mathematics came to be perceived as a diligent and unquestioning appli-

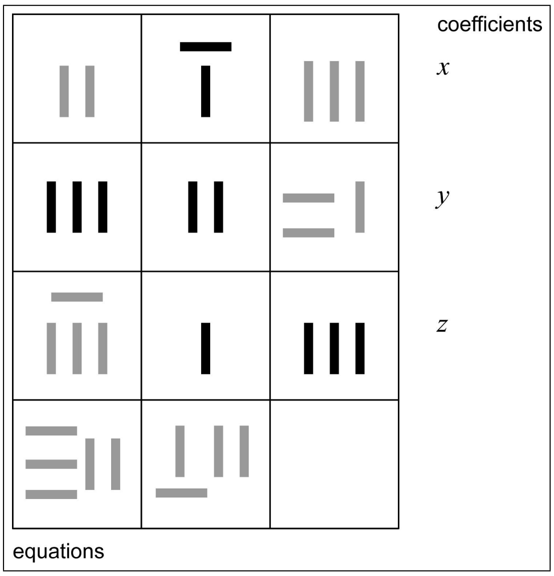

Figure 2. Representing simultaneous equations with counting rods. Red rods (here shown in grey) indicate positive coefficients, black rods negative coefficients. Redrawn, with permission, from Ref. 23, p. 146, Fig. 4.6.

cation of ancient wisdom, rather than an ongoing creative process [23]. Chinese mathematics never became an autonomous discipline: mathematicians remained technical experts dealing with chronology, finances, taxation, architecture, and the military [24].

3.2. *Islamic Algebra*

Islamic algebra developed as a combination of and elaboration upon cultural inventions borrowed from other traditions, notably Greek geometry, Indian numerals and arithmetic, and Nestorian astronomy. This was possible because of the patronage of the Abbasid caliphate (758–1258), centred in Baghdad. It commissioned translations of major works by Aristotle, Galen, and many Persian and Indian scholars into Arabic. This knowledge became appropriated and reconstructed in Islamic mathematics [27]. It built upon a combination of Greek geometric visualization and 'proofs', and Indian and Chinese arithmetic rules to solve equations quickly and efficiently. Through this eclectic approach, Arabian mathematicians could formulate solutions that they could not find through arithmetic alone. When, for example, Omar Khayyám (1048–1123) failed to find an arithmetic solution to cubic equations, he resorted to a geometric method of intersecting cones to solve them [23]. After the fourteenth century, Islamic mathematics and other sciences apparently stagnated and declined. A likely cause for this dwindling was the growing incompatibility between Islamic metaphysics and science. According to the Qu'ran, the world is governed by Allah's uninterrupted control of all events. This line of thought inhibits the possibility of a world governed by rational, knowable and coherent laws of nature, being a principle of Greek philosophy. Nevertheless, between the eighth and eleventh century, Islam did adopt scientific thought in the form of philosophy, astronomy, medicine and mathematics. However, influential theologians such as al-Ash'ari (873–935) and al-Ghazali (1058–1111) questioned the validity of Greek philosophy which they saw as a menace to religious orthodoxy. They became committed to Islamic occasionalism. According to this doctrine, the basic building blocks of nature are individual indivisible 'atoms', each of which lasts only a moment and then disappears. Allah creates the world anew each moment of time, thereby giving it pattern and persistence. The world by itself cannot exist without this continuous creation and is therefore not ruled by laws of nature. Such a worldview is deeply incompatible with the basic assumptions of science that the world is governed by rational, coherent and orderly laws [28]. Mathematics and its application in astronomy and other branches of science did not escape this devastating attack. The fourteenth century theologian al-Iji wrote that "we can disregard mathematical entities as they are more tenuous than a spider's web". From the tenth century onwards, occasionalism became the dominant orthodoxy—incidentally, it still is. Hampered by religious orthodoxy, mathematics and other sciences could no longer flourish, with the exception of some applied mathematics that had highly political or religious importance, such as the construction of a round cupola on the square basis of mausoleums for political rulers, or the calculation of the qibla, the relative position of Mecca [27]. Like in China, mathematics never became an autonomous discipline, but remained ancillary to engineering, cartography, optics and geography.

3.3. *European Algebra*

European algebra starts in the late thirteenth century commercial cities of the Catalonian, Provençal and northern Italian regions among mathematicians with a background in teaching commercial calculation methods. However, while showing substantial influence from Islamic mathematics, this algebra went beyond it in its ambition to solve irreducible higher-degree problems, starting with Jacopo da Firenze's *Tractatus algorismi* (1307) [29]. Since the seventeenth century, mathematicians began to work at increasingly abstract problems, including algebraic series and the calculus, culminating in collective conflicts among synthesizing schools with rival programs on the foundations and nature of mathematics.

A disparate number of cultural factors have jointly contributed to this evolution. In contrast to Islamic metaphysics, Christian metaphysics endorsed the notions of natural laws and causality. The Scholastics saw evidence of God's harmonious creation, fully in line with Plato's system of nature, as a system of causal necessity. The metaphor of this *machina mundi* is ubiquitous in their writings. The arrival of Arabic translations of Aristotle in the thirteenth century gave this enthusiasm for naturalistic enquiry another powerful boost. Although modern science has traditionally been seen as a child of the Enlightenment, this development has actually been more gradual. In particular, the emergence of Nominalism within Scholasticism in the second half of the thirteenth century, was an important factor in the development of science. The Nominalists introduced empirical investigation, arguing that we can only discover the world by observation [30]. Medieval universities institutionalized philosophical enquiry, making it acceptable that knowledge acquired through scientific investigation, became a valid source of knowledge next to the Scriptures [28]. Thus, seventeenth century science can be seen as a second phase in an intellectual movement that began in the thirteenth century. Nevertheless, it was not until the seventeenth century that a fundamental shift in the perception of science enabled turnovers such as the transformation from Ptolemaic to Keplerian astronomy. Medieval mathematicians believed that mathematics had deep roots in classical Greece: there always remained an unchanging body of knowledge which could at best be preserved and at worst corrupted by contemporary scholars. Gradually, however, they came to realize that mathematicians could elaborate and even improve upon this body of knowledge. In this newly emerged paradigm, science was no longer perceived as ancient revealed knowledge, but as a human endeavour influenced by culture and circumstance. For example, John Wallis, a seventeenth century historian of mathematics, not only wrote that the Muslims had developed and improved classical texts, but proudly stated the further achievements of his contemporaries [31]. European mathematicians thus became increasingly aware that they themselves created the levels of abstraction in mathematics. This realization led to many innovations, such as the introduction of imaginary numbers. Take i, which symbolizes an operation that cannot actually be performed, namely extracting a square root from a negative number. By denoting this operation it becomes possible to integrate roots from negative numbers in operations, thus vastly extending the scope of mathematics. Only European mathematics has consistently aimed at representing such opaque operations by a constant set of symbols. Indian mathematicians, for example, used the symbol 0 for both zero and the unknown in equations [23]. Such ambiguities, or in the case of Chinese algebra, a lack of external symbols for operations, inevitably limit the degree of abstraction in mathematics.

4. The Extended Mind and the Autonomy of Mathematical Thought

External symbols can provide anchors for thoughts that are difficult to understand or represent. Without them, such thoughts would not survive long in the competition for attention and for cognitive resources that characterizes cultural evolution. Indeed, theoretical models that examine cultural representations from an epidemiological perspective predict that concepts that are hard to learn and hard to represent are quickly outcompeted in favour of intuitive or attention-grabbing ideas. These predictions have been experimentally confirmed in a variety of cultural settings. For example, story recall experiments illustrate that both Hindu [32] and Christian [33] college students do not intuitively think about their respective gods as their theologies require. Christians have difficulties representing God as an omniscient, omnipresent being and distort stories about him to fit intuitive expectations they have about normal people, like that he can only attend to one person or one event at the same time. External representations can anchor non-intuitive concepts on a more permanent basis. Seeing mathematical symbols facilitates the recall of previously stored knowledge. Once mathematical concepts are nested outside of the brain, their evolution and cultural transmission is less vulnerable to corruption by individual mathematicians or to competition from ideas that are easier to learn, that speak more to the imagination, or that pose less computational demands. They gain a degree of autonomy that would be impossible to attain were they represented in the mind alone. In such distributed cognitive systems [34] both the interaction of human minds with external notation systems, and the interaction of many mathematicians' minds enhance the cognitive capacities of the brain.

The consistent striving of European mathematicians to externalize operations has given mathematics a unique representational power. This 'unreasonable' efficiency of mathematics as an epistemic tool is, I argue, an emergent property of its hybrid structure. Since mathematicians routinely express operations that can never be performed in the mind alone (e.g. a negative number, which results from the subtraction of a larger number from a smaller one), mathematics can convey a range of ideas that are actually opaque. Let us examine this distinction between transparent and opaque more closely. A transparent concept is one to which we have semantic access; we intuitively grasp its meaning. In mathematics, Arabic numerals which denote positive integers (1, 2, 3, ...) are transparent concepts. Studies of brain activation in adults and even five-year-olds have shown that a number comparison task with Arabic numerals (say 5 and 3) activates the same brain areas as one that involves arrays of dots (say five dots and three dots)—even the speed of computation is identical [35]. Thus, the brain immediately translates a positive integer into a mental representation of its quantity. In contrast, when confronted with i or -3, no such translation takes place, because these operations cannot actually be performed in the mind (it is impossible to imagine a negative quantity). The only way to represent them is through an external set of symbols, the meaning of which remains semantically inaccessible to us. Through this externalization, we expand our representational abilities by delegating operations that are impossible to perform in the brain to our external environment, by denoting them through a symbolic system. Mathematical concepts, in themselves a hybrid structure composed of transparent and non-transparent symbolic notations, can in its turn become an epistemic tool. We denote processes in nature through mathematical formulations, because the latter have no clear semantic content. Therefore, identical functions and equations can be used in disparate contexts, including physics, biology, and anthropology.

5. Conclusion

In this paper, I have argued that algebra has emergent properties, resulting from the co-optation of evolved mental abilities and the externalisation of operations in a symbolic system. Such a system needs extensive cultural support, both from the metaphysics of the culture it belongs to and the ontological status of mathematics. It is crucial for mathematics and other sciences that a metaphysical view is endorsed in which the world is governed by causal laws, and in which humans are capable of acquiring reliable knowledge of the world. Once mathematicians realized their practice is a human endeavour, and that they can improve it, they could create higher levels of abstraction by symbolically representing operations. Early modern European mathematics was not superior to its Chinese and Arabian contemporaries—rather, it could evolve because cultural conditions were more favourable.

Acknowledgements

I gratefully acknowledge helpful comments by Johan De Smedt and Jean Paul Van Bendegem on an earlier draft. This research is supported by grant OZR916BOF from the Free University of Brussels.

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